A Structural Model for Credit Migrations

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Credit Ratings and Transition Matrices

- Ratings are conducted using traditional actuarial methods by rating agencies, mainly Standard & Poors, Moody’s, Fitch, and Duff & Phelps.
- Transition matrices are estimated based on actuarial methods. From these matrices, estimated default probabilities (default rates) are inferred.
- They are used in JP Morgan’s Credit Metrics and McKinsey’s Credit Portfolio View, … etc.
# Credit Ratings

<table>
<thead>
<tr>
<th>Explanation</th>
<th>Standard &amp; Poors</th>
<th>Moody's</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Investment grade</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Highest grade</td>
<td>AAA</td>
<td>Aaa</td>
</tr>
<tr>
<td>High grade</td>
<td>AA</td>
<td>Aa</td>
</tr>
<tr>
<td>Upper medium grade</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>Medium grade</td>
<td>BBB</td>
<td>B aa</td>
</tr>
<tr>
<td><strong>Speculative grade</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower medium grade</td>
<td>BB</td>
<td>Ba</td>
</tr>
<tr>
<td>Speculative</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>Poor standing</td>
<td>CCC</td>
<td>Caa</td>
</tr>
<tr>
<td>High speculative</td>
<td>CC</td>
<td>Ca</td>
</tr>
<tr>
<td>Lowest quality, no interest</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>In default</td>
<td>D</td>
<td></td>
</tr>
</tbody>
</table>
# Credit Migration

## Transition Matrix

<table>
<thead>
<tr>
<th>Initial rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>93.66</td>
<td>5.83</td>
<td>0.40</td>
<td>0.09</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>AA</td>
<td>0.66</td>
<td>91.72</td>
<td>6.94</td>
<td>0.49</td>
<td>0.06</td>
<td>0.09</td>
<td>0.02</td>
<td>0.01</td>
</tr>
<tr>
<td>A</td>
<td>0.07</td>
<td>2.25</td>
<td>91.76</td>
<td>5.18</td>
<td>0.49</td>
<td>0.20</td>
<td>0.01</td>
<td>0.04</td>
</tr>
<tr>
<td>BBB</td>
<td>0.03</td>
<td>0.26</td>
<td>4.83</td>
<td>89.24</td>
<td>4.44</td>
<td>0.81</td>
<td>0.16</td>
<td>0.24</td>
</tr>
<tr>
<td>BB</td>
<td>0.03</td>
<td>0.06</td>
<td>0.44</td>
<td>6.66</td>
<td>83.23</td>
<td>7.46</td>
<td>1.05</td>
<td>1.08</td>
</tr>
<tr>
<td>B</td>
<td>0.00</td>
<td>0.10</td>
<td>0.32</td>
<td>0.46</td>
<td>5.72</td>
<td>83.62</td>
<td>3.84</td>
<td>5.94</td>
</tr>
<tr>
<td>CCC</td>
<td>0.15</td>
<td>0.00</td>
<td>0.29</td>
<td>0.88</td>
<td>1.91</td>
<td>10.28</td>
<td>61.23</td>
<td>25.26</td>
</tr>
<tr>
<td>Default</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>100.00</td>
</tr>
</tbody>
</table>

*Source: Standard & Poor’s, January 2001.*

Migrating probability from BBB to A
Traditional Ratings

- Traditional rating method relies on analyzing financial statements. Rating transition probabilities are based on average historical frequencies of defaults and credit migrations. One assumes that actual default rates and credit changes equal to their historical counterparts.

- In practice, credit migrations and default rate changes are taken to be equivalent. However, default rates evolve continuously, but credit migrations are reported in a discrete fashion.

- Traditional rating method classifies firms into credit categories. But how do the rating agencies classify firms according to default probabilities?
Kealhofer et. al (1998) found a high degree of overlaps of estimated default probabilities across different letter grades according to the S&P ratings. In addition to EDP, traditional ratings incorporate other information to rate a firm. But what and how? The structural model (option theoretic) approach used by KMV monitors the default probability continuously. In principle, one could rate each company based on default probabilities. But this is difficult to implement. Moody’s acquired KMV in 2002. It now becomes feasible to examine credit migration from the option-theoretic point of view.
Let $V$ denote the value of the firm’s asset, $B$ the market value of the loan, and $F$ the face value of the loan at maturity $T$.

Credit risk exists as long as $P(V_T < F) > 0$.

What can a bank do to eliminate the credit risk?

Bank longs a put option on the asset value (stochastic) of the obligor, at a strike price $F$, maturing at time $T$.

If the bank purchases such an option, it would eliminate the credit risk associated with the loan completely.
The bank’s payoff matrix at times 0 and T with the purchase of the put option.

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of assets</td>
<td>$V_0$</td>
<td>$V_T \leq F$ &lt;br&gt; $V_T &gt; F$</td>
</tr>
<tr>
<td>Bank’s Position: (a) Make a loan</td>
<td>$-B_0$</td>
<td>$V_T$ &lt;br&gt; $F$</td>
</tr>
<tr>
<td>(b) Long a put</td>
<td>$-P_0$</td>
<td>$F-V_T$ &lt;br&gt; $0$</td>
</tr>
<tr>
<td>Total</td>
<td>$-B_0 - P_0$</td>
<td>$F$ &lt;br&gt; $F$</td>
</tr>
</tbody>
</table>
The value of the put is the cost of eliminating the credit risk, which by means of the Black-Scholes formula is a function of the default probability.

One is then interested in the default probability (DF), \( P(V_T < F) \).

The KMV approach derives the estimated default probability (EDP) based on Merton’s structural model approach.

Assume the firm asset value \( V_t \) follows a geometric Brownian motion. Then the distance to default is defined as:
KMV Approach

- Distance to default, $DD_t$:

\[ X_t = \frac{\ln V_t - \ln F_t}{\sigma_{V_t}} , \]

where $F_t = CL_t + LTD_t / 2$ is the default point.

An alternate definition of $DD$ is

\[ DD_t = \frac{\text{Firm Asset Value} (V_t) - \text{Default Point} (F_t)}{\text{Firm Asset Value} (V_t) \times \text{Asset Volatility} (\sigma_{V_t})} \]
EDPs

- No inter-temporal default
  \[ EDP_1(t, T) = P(X_T \leq 0 \mid X_t) \]

- Inter-temporal default
  \[ EDP_2(t, T) = P(\min_{t \leq s \leq T} X_s \leq 0 \mid X_t) \]
Diffusion model

- Under this model, $X_t$ follows a Brownian motion
  
  \[ dX_t = \mu dt + \sigma dW_t, \]

  where $\mu$ and $\sigma$ are constants.

- EDPs become:

  \[
  EDP_1(t, T) = N\left(-\frac{X_t + \mu(T-t)}{\sigma \sqrt{T-t}}\right)
  \]

  \[
  EDP_2(t, T) = N\left(-\frac{X_t + \mu(T-t)}{\sigma \sqrt{T-t}}\right) + e^{\frac{2\mu X_t}{\sigma^2}} N\left(\frac{X_t - \mu(T-t)}{\sigma \sqrt{T-t}}\right)
  \]
Credit Migration

Credit Classes

\[ X_t \]

\[ C_0 = \text{Default Grade} \]

\[ C_1 \]

\[ C_2 \]

\[ C_n = \text{AAA} \]

\[ 0 = B_0 \]

\[ B_1 \]

\[ B_2 \]

\[ B_{n-1} \]

\[ B_n = M \]
Migrating Signals

- $X_t =$ Distance to Default
- $c_t$ is the credit rating variable; $c_t = k$ if the firm is rated in class $k$ (like a BB-firm) at time $t$.
- Migrating signals: $X_t \not\in (B_{k-1}, B_k]$ when $c_t = k$
  - Upgrade signal: If $X_t > B_k$ given $c_t = k$
  - Downgrade signal: If $X_t \leq B_{k-1}$ given $c_t = k$
- Migrating signal Durations:
  - Upgrading duration ($\Gamma_k^+$): the total time of an upgrading signal.
  - Downgrading duration ($\Gamma_k^-$): the total time of a downgrading signal.
Credit Migration

Mathematically,

\[ \Gamma^+_k(t,T) = \int_t^T I(X_s > B_k) \, ds \]

\[ \Gamma^-_k(t,T) = \int_t^T I(X_s \leq B_{k-1}) \, ds \]

where \( I(\bullet) \) is the indicator function.

Migration Probability:

\[ p_{kj}(0,T) = P(c_T = j \mid c_0 = k) \]
Credit Migration

Migrating Flow Chart

- Upgrade to class $j$ if $X_T \in (B_{j-1}, B_j]$.
- Downgrade to class $j$ if $X_T \in (B_{j-1}, B_j]$.

1. Showing a migrating signal? ($X_T \in (B_{k-1}, B_k]$?)
   - Upgrade Signal
     - $\Gamma_k^+ \not\rightarrow \alpha_{k\rightarrow j} T^?$
     - Yes
       - Upgrade to class $j$ if $X_T \in (B_{j-1}, B_j]$.
     - No
       - No
       - Unchanged
   - Downgrade Signal
     - $\Gamma_k^- \not\rightarrow \alpha_{k\rightarrow j} T^?$
     - Yes
       - Downgrade to class $j$ if $X_T \in (B_{j-1}, B_j]$.
     - No
       - No
       - Unchanged
Interpretation of $\alpha_{k->j}$

- A highly rated firm enjoys the privilege that it would not be downgraded due to a short term decrease in credit quality. ($\Gamma_k^- < \alpha_{k->j} T, j < k$).

- A poorly rated firm needs to create a positive credit profile in order to be upgraded. ($\Gamma_k^+ > \alpha_{k->j} T, j > k$).

- $\alpha_{k->0} = 0$ because the firm has no reputation when defaults.
Migrating Probability

- Assumptions:
  - $X_t$ follows a Brownian motion:
    \[ dX_t = \mu dt + \sigma dW_t \]
  - No inter-temporal default in the period $(0, T)$, i.e. the firm can only default at $t = T$.

\[
p_{k0}^{1}(0,T) = P(c_T = 0 \mid c_0 = k) = P(X_T \leq 0, \Gamma_k > 0 \mid X_0) \\
= P(X_T \leq 0 \mid X_0) = EDP_1.
\]

- To evaluate the migrating probability, one needs the joint density function, $\psi$, of $(X_t, \Gamma_k)$. 
Occupation Time density

- The joint density function of \((X_t, \Gamma_k^-)\) satisfies the Fokker-Planck equation:

\[
\frac{\partial \psi}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial x^2} - \mu \frac{\partial \psi}{\partial x} - I(x < B_{k-1}) \frac{\partial \psi}{\partial \Gamma_k^-} = 0 \quad \text{for } t > 0, \Gamma_k^- > 0
\]

\[
\psi \bigg|_{t=0} = \delta(x)\delta(\Gamma_k^-),
\]

\[
\psi \bigg|_{\Gamma_k^- = 0} = P(x \in dx, \min_{i \leq s \leq T} X_s > B_{k-1})
\]

- The solution \(\psi\) can be found in Linetsky (1999).

- \(p_{kj}(0, T) = E \{I(X_T \in (B_{j-1}, B_j)) I(\Gamma_k^- > \alpha_{k\rightarrow j} T) | X_0\}\) for \(j < k\).
Downgrading Probability

- Downgrading probability:
  If \( X_0 < B_{k-1} \), then
  \[
p_{kj}(0, T) = N \left( \frac{y_2 - x - \mu T}{\sigma \sqrt{T}} \right) - N \left( \frac{y_1 - x - \mu T}{\sigma \sqrt{T}} \right)
  \]
  \[- e^{-2\frac{\mu x}{\sigma^2}} \left[ N \left( \frac{y_2 + x - \mu T}{\sigma \sqrt{T}} \right) - N \left( \frac{y_1 + x - \mu T}{\sigma \sqrt{T}} \right) \right]
  \]
  \[+ \int_0^T F(t, T)[h(y_1, x, t) - h(y_2, x, t)] dt.\]

  If \( X_0 \geq B_{k-1} \), then
  \[
p_{kj}(0, T) = \int_0^T F(t, T)[g(y_1, x, t) - g(y_2, x, t)] dt.
  \]
Credit Migration

Upgrading Probability

- Upgrading probability:
  - If $X_0 \geq B_k$, then
    \[
    p_{kj} (0, T) = N \left( \frac{z_2 - \hat{x} - \mu T}{\sigma \sqrt{T}} \right) - N \left( \frac{z_1 - \hat{x} - \mu T}{\sigma \sqrt{T}} \right) \\
    - e^{- \frac{2 \mu \hat{x}}{\sigma^2}} \left[ N \left( \frac{z_2 + \hat{x} - \mu T}{\sigma \sqrt{T}} \right) - N \left( \frac{z_1 + \hat{x} - \mu T}{\sigma \sqrt{T}} \right) \right] \\
    + \int_0^T F(t, T) \left[ h(z_2, \hat{x}, t) - h(z_1, \hat{x}, t) \right] dt.
    \]
  - If $X_0 < B_k$, then
    \[
    p_{kj} (0, T) = \int_0^T F(t, T) \left[ g(z_2, \hat{x}, t) - g(z_1, \hat{x}, t) \right] dt.
    \]
Fundamental Equations

\[ F(t, T) = \begin{cases} (1 - \alpha)T, & 0 \leq t \leq \alpha T \\ T-t, & \alpha T < t \leq T \end{cases} \]

\[ g(y, x, t) = \frac{n \left( \frac{x + \mu (T-t)}{\sigma \sqrt{T-t}} \right)}{(T-t)^{3/2}} \left[ D_1 n \left( \frac{y - \mu t}{\sigma \sqrt{t}} \right) + D_2 N \left( \frac{y - \mu t}{\sigma \sqrt{t}} \right) \right], \]

\[ D_1 = \frac{2x\mu}{\sigma^2} - \left(1 - \frac{x^2}{\sigma^2 (T-t)}\right) \sqrt{t} + \frac{x(y - \mu t)}{\sigma^2 \sqrt{t}}, \]

\[ x = X_0 - B_{k-1}, \; \hat{x} = X_0 - B_k, \]

\[ y_1 = B_{j-1} - B_{k-1} \quad \text{and} \quad y_2 = B_j - B_{k-1}, \]

\[ z_1 = B_{j-1} - B_k \quad \text{and} \quad z_2 = B_j - B_k, \]

\[ h(y, x, t) = \frac{e^{-2\mu x/\sigma^2 + \mu^2 (T-t)}}{\sqrt{2\pi} (T-t)^{3/2}} \left[ \mu N \left( \frac{y + x - \mu t}{\sigma \sqrt{t}} \right) - \frac{1}{2\sqrt{t}} n \left( \frac{y + x - \mu t}{\sigma \sqrt{t}} \right) \right]. \]
Applications

- Values of $\alpha_{k->j}$ for each $k$ with $j \neq k$ can be approximated through a proxy method if the drift, the volatility, and the transition matrix are given. This value tells us for how long should a class $k$ firm perform well to be upgraded (downgraded) to class $j$ for $j > k$ ($j < k$).

- Conversely, given $\alpha_{k->j}$, the drift $\mu$ and the volatility $\sigma$ of the Distance to Default, $X_t$, the transition vector for a specific firm can be obtained.
Example: Evaluation of $\alpha_{k->j}$

Consider the following transition matrix from S&P (1981-2000):

<table>
<thead>
<tr>
<th>Initial rating</th>
<th>A A A</th>
<th>A A</th>
<th>A</th>
<th>B B B</th>
<th>B B</th>
<th>B</th>
<th>C C C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A A A</td>
<td>93.66</td>
<td>5.83</td>
<td>0.4</td>
<td>0.09</td>
<td>0.03</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>A A</td>
<td>0.66</td>
<td>91.72</td>
<td>6.94</td>
<td>0.49</td>
<td>0.06</td>
<td>0.09</td>
<td>0.02</td>
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<td>0.07</td>
<td>2.25</td>
<td>91.76</td>
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<td>0.49</td>
<td>0.2</td>
<td>0.01</td>
<td>0.04</td>
</tr>
<tr>
<td>B B B</td>
<td>0.03</td>
<td>0.26</td>
<td>4.83</td>
<td>89.24</td>
<td>4.44</td>
<td>0.8</td>
<td>0.15</td>
<td>0.24</td>
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<tr>
<td>B B</td>
<td>0.03</td>
<td>0.06</td>
<td>0.44</td>
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<td>83.23</td>
<td>7.46</td>
<td>1.04</td>
<td>1.08</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0.1</td>
<td>0.32</td>
<td>0.46</td>
<td>5.72</td>
<td>83.62</td>
<td>3.84</td>
<td>5.94</td>
</tr>
<tr>
<td>CCC</td>
<td>0.15</td>
<td>0</td>
<td>0.29</td>
<td>0.88</td>
<td>1.91</td>
<td>10.28</td>
<td>61.23</td>
<td>25.26</td>
</tr>
</tbody>
</table>
Assume $\mu = 0$ and $\sigma = 1$. Recall

$$EDP_1(t, T) = N\left( -\frac{X_t + \mu(T - t)}{\sigma \sqrt{T - t}} \right)$$

- If $c_0 = k$, then a typical default distance is given by $X_0^{(k)} = -N^{-1}$ (Default probability of class $k$)

- Let the boundary $B_k = (X_0^{(k)} + X_0^{(k+1)})/2$. The range of Distance to Default, $(B_{k-1}, B_k]$, of each credit class can then be constructed.

- With given $\mu$, $\sigma$, and the transition probability of credit migration, $p_{kj}(0, T)$, use the fundamental equations to compute values of $\alpha_{k->j}$ numerically.

- From the computed $B_k$, one can calculate the EDPs by the formula $EDP = N(-B_k)$.

- Results are given in the following table.
<table>
<thead>
<tr>
<th>Rating</th>
<th>EDP range (%)</th>
<th>((B_{k-1}, B_k])</th>
<th>(X_0^{(k)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.0035 – 0.007</td>
<td>3.9764 – 3.8048</td>
<td>3.8906</td>
</tr>
<tr>
<td>AA</td>
<td>0.007 – 0.023</td>
<td>3.8048 – 3.5359</td>
<td>3.7190</td>
</tr>
<tr>
<td>A</td>
<td>0.023 – 0.101</td>
<td>3.5359 – 3.0865</td>
<td>3.3528</td>
</tr>
<tr>
<td>BBB</td>
<td>0.101 – 0.5253</td>
<td>3.0865 – 2.5587</td>
<td>2.8202</td>
</tr>
<tr>
<td>BB</td>
<td>0.5253 – 2.689</td>
<td>2.5587 – 1.9268</td>
<td>2.2973</td>
</tr>
<tr>
<td>B</td>
<td>2.689 – 13.28</td>
<td>1.9268 – 1.1131</td>
<td>1.5598</td>
</tr>
<tr>
<td>CCC</td>
<td>13.28 – 30.86</td>
<td>1.1131 – 0.4997</td>
<td>0.6663</td>
</tr>
</tbody>
</table>
Computed values of $\alpha_{k->j}$

<table>
<thead>
<tr>
<th></th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>Nil</td>
<td>0.964</td>
<td>0.995</td>
<td>0.997</td>
<td>0.998</td>
<td>0.998</td>
<td>0.997</td>
</tr>
<tr>
<td>AA</td>
<td>0.995</td>
<td>Nil</td>
<td>0.906</td>
<td>0.983</td>
<td>0.989</td>
<td>0.987</td>
<td>0.987</td>
</tr>
<tr>
<td>A</td>
<td>0.991</td>
<td>0.968</td>
<td>Nil</td>
<td>0.879</td>
<td>0.968</td>
<td>0.974</td>
<td>0.977</td>
</tr>
<tr>
<td>BBB</td>
<td>0.988</td>
<td>0.978</td>
<td>0.892</td>
<td>Nil</td>
<td>0.888</td>
<td>0.958</td>
<td>0.963</td>
</tr>
<tr>
<td>BB</td>
<td>0.987</td>
<td>0.984</td>
<td>0.973</td>
<td>0.882</td>
<td>Nil</td>
<td>0.685</td>
<td>0.893</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>0.959</td>
<td>0.946</td>
<td>0.940</td>
<td>0.776</td>
<td>Nil</td>
<td>0.531</td>
</tr>
<tr>
<td>CCC</td>
<td>1</td>
<td>1</td>
<td>0.811</td>
<td>0.812</td>
<td>0.813</td>
<td>0.629</td>
<td>Nil</td>
</tr>
</tbody>
</table>
Observations

- Consider an A-rated firm. Based on a one year credit profile, an A-firm would be downgraded to BBB lower within a year only when it shows a downgrade signal more than ten months.

- On the other hand, it has to perform well for most of the year (11.9 months) to be upgraded to AAA within a year.
### Example: Overlaps of EDPs

<table>
<thead>
<tr>
<th>Ratings</th>
<th>EDP range</th>
<th>$X_t$-range ($B_{k-1}$, $B_k$)</th>
<th>No of firms</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.0035 – 0.007</td>
<td>3.9764 – 3.8048</td>
<td>$n_1$</td>
</tr>
<tr>
<td>AA</td>
<td>0.007 – 0.023</td>
<td>3.8048 – 3.5359</td>
<td>$n_2$</td>
</tr>
<tr>
<td>A</td>
<td>0.023 – 0.101</td>
<td>3.5359 – 3.0865</td>
<td>$n_3$</td>
</tr>
<tr>
<td>BBB</td>
<td>0.101 – 0.5253</td>
<td>3.0865 – 2.5587</td>
<td>$n_4$</td>
</tr>
<tr>
<td>BB</td>
<td>0.5253 – 2.689</td>
<td>2.5587 – 1.9268</td>
<td>$n_5$</td>
</tr>
<tr>
<td>B</td>
<td>2.689 – 13.28</td>
<td>1.9268 – 1.1131</td>
<td>$n_6$</td>
</tr>
<tr>
<td>CCC</td>
<td>13.28 – 30.86</td>
<td>1.1131 – 0.4997</td>
<td>$n_7$</td>
</tr>
<tr>
<td>NR(Not rated)</td>
<td>30.86 – 50</td>
<td>0.4997 – 0.0</td>
<td>$n_8$</td>
</tr>
<tr>
<td>D</td>
<td>50 – 100</td>
<td>0.0 – (−∞)</td>
<td>0</td>
</tr>
</tbody>
</table>

$\alpha = 0.8$ for upgrade and $\alpha = 0.5$ for downgrade
At $t = 0$, simulate the Distance to Default, $X_0^{(i)}$ according to the class variable $c_0^{(i)}$ of each firm as follows, $i = 1, ..., n_k$. If $c_0^{(i)} = k$, simulate the 1-year $X_0^{(i)}$ from $U(-B_k, -B_{k-1})$.

Simulate a five-year sample path of $X$ for each firm by the model $X_{t+\Delta_t}^{(i)} = X_t^{(i)} + \varepsilon \sqrt{\Delta t}$, $\varepsilon \sim N(0, 1)$, $t = 1, ..., T$.

At the end of each year, the values of $c_t^{(i)}$ are reviewed according to the proposed criterion, for all $i = 1, 2, ..., n_k$.

Given $c_t^{(i)} = k$, calculate the one year ahead EDP value of firm $i$ by $\text{EDP}(i) = N(-X_T^{(i)})$, where $X_T^{(i)}$ is the simulated distance to default value of firm $i$ at the end of the 5th year.
Results
Future Directions

- Calibrate bounds for each credit class. Two possible methods are:
  - Extending the method of calibrating default boundaries.
  - Using CART approach.
- Extend to inter-temporal default.
- Consider the jump-diffusion economy.
- Conduct statistical tests for the model.
Future Events

- Workshop in Risk Management on July 18, 2003 in Conrad Hotel.
- Two year part-time Master of Science Program in Risk Management starting this fall at CUHK.
Credit Migration

Transition Matrix

Transition Matrix: Probabilities of Credit Rating Migrating From One Rating Quality to Another, Within One Year

<table>
<thead>
<tr>
<th>Initial Rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>90.81</td>
<td>8.33</td>
<td>0.68</td>
<td>0.06</td>
<td>0.12</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AA</td>
<td>0.70</td>
<td>90.65</td>
<td>7.79</td>
<td>0.64</td>
<td>0.06</td>
<td>0.14</td>
<td>0.02</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>0.09</td>
<td>2.27</td>
<td>91.05</td>
<td>5.52</td>
<td>0.74</td>
<td>0.26</td>
<td>0.01</td>
<td>0.06</td>
</tr>
<tr>
<td>BBB</td>
<td>0.02</td>
<td>0.33</td>
<td>5.95</td>
<td>86.93</td>
<td>5.30</td>
<td>1.17</td>
<td>1.12</td>
<td>0.18</td>
</tr>
<tr>
<td>BB</td>
<td>0.03</td>
<td>0.14</td>
<td>0.67</td>
<td>7.73</td>
<td>80.53</td>
<td>8.84</td>
<td>1.00</td>
<td>1.06</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0.11</td>
<td>0.24</td>
<td>0.43</td>
<td>6.48</td>
<td>83.46</td>
<td>4.07</td>
<td>5.20</td>
</tr>
<tr>
<td>CCC</td>
<td>0.22</td>
<td>0</td>
<td>0.22</td>
<td>1.30</td>
<td>2.38</td>
<td>11.24</td>
<td>64.86</td>
<td>19.79</td>
</tr>
</tbody>
</table>

Source: Standard & Poor's CreditWeek (April 15, 1996).

Migrating probability from BBB to A
This model matches the intuition that transition to state 0 is the same as the probability of default. Since $\alpha_{k->0} = 0$,

$$p_{k0}^1(0, T) = P(c_T = 0 \mid c_0 = k) = P(X_T \leq 0, \Gamma_k^- > 0 \mid X_0)$$

$$= P(X_T \leq 0 \mid X_0) = EDP_1,$$

$$p_{k0}^2(0, T) = P(c_T = 0 \mid c_0 = k) = P(\min_{0 \leq s \leq T} X_s \leq 0, \Gamma_k^- > 0 \mid X_0)$$

$$= P(\min_{0 \leq s \leq T} X_s \leq 0 \mid X_0) = EDP_2,$$

This model allows for possible overlaps in EDP across different credit classes (letter grades).
Overlaps in EDP for different ratings

EDP (EDF), Expected Default Probability (Frequency)

Credit Migration

\[ X_t \]

Downgrade duration

Upgrade duration

May upgrade to \( C_n \)

N.H. Chan